MOTION OF DEFLECTED FRONT OF A SHOCK WAVE IN A NONUNIFORM MEDIUM

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The problem of front stability of a shock wave expanding in the direction of lower densities of the medium is an interesting problem for a very wide class of nonlinear motions: For example, phenomena such as atmospheric outburst [1] or turbulization of a gas cloud and of dust following nuclear explosions in the atmosphere [2], the emergence of large-scale turbulence in envelopes of blazing stars of the supernova type [3], and front propagation with gradiental accelerations in laboratory investigations [4] are in one way or another related with lack of stability of supersonic motions. Theoretical analysis has shown that the front of a strong shock wave (of Mach number $M \gg 1$) propagated in a medium of decreasing density is unstable. The analysis was carried out in [5, 6] in the case of no magnetic field and in the case of a magnetohydrodynamic shock wave in a linear approximation with regard to perturbations. Random deflections in the front, in which separate elements are ahead of the front or trail it, grow with time; the latter proves to be true even if one uses the nonlinear corrections applied in this article if there is no magnetic field. The evolution of small distortions of the front of a shock wave propagated in a uniform gas was considered in [7] and in more detail (for arbitrary state equation of the gas) in [8]. In the case of a strong shock wave the results obtained in these articles follow directly from the results obtained in the present article and show that in a uniform medium the front remains stable for known state equations.

§1. Let an unperturbed plane front of a strong shock wave move along the normal in the positive direction of the x axis, and let the density of the unperturbed gas $\rho_0(X)$ decrease in the same direction. In the cases of either exponential or power laws for the density drop, the front velocity $u(X)$ and its position $X(t)$ can be determined by numerical integration of the equations containing the self-consistent variable. Approximate values of these quantities can be found using the Chisnell–Whitham method (see, for example, [9]). Considerable improvements to this method were obtained in [10]; it was shown that

$$u \sim \rho_0^{-\lambda}, \quad \lambda = 2 + \frac{\gamma+1}{2}\sqrt{\frac{2\gamma}{\gamma-1}},$$

where γ is the adiabatic index of the matter, whereas in [9] the index was given by $\lambda_0 = 2 + \sqrt{2\gamma/(\gamma-1)}$.

The method as well as the notation used in [5, 6, 10] will be used in our further considerations. However, to develop further the method, as well as to obtain nonlinear corrections, the front perturbations, in contrast to the above-mentioned methods, are not assumed at first to be small. Let the coordinate of the perturbed-front region be $\Xi(y, t) = X(t) + \xi(y, t)$, and its angular deviation from the plane one be $\theta = \arctan(d\xi/dy)$. The origin is now transferred to the point $\dot{\xi}$ and the coordinate axes are rotated in such a way that y' is tangential to the front. When transferring to the new variables, the derivatives which appear in the equations of hydrodynamics are replaced in the following manner:

$$\frac{\partial}{\partial x} = \cos\theta\frac{\partial}{\partial x'} + \sin\theta\frac{\partial}{\partial y'}; \quad \frac{\partial}{\partial y} = \cos\theta\frac{\partial}{\partial y'} - \sin\theta\frac{\partial}{\partial x'},$$
$$\frac{\partial}{\partial t} = \frac{\partial}{\partial t} - \dot{\Xi}\left(\cos\theta\frac{\partial}{\partial x'} + \sin\theta\frac{\partial}{\partial y'}\right) - \dot{\theta}\left(y'\frac{\partial}{\partial x'} - x'\frac{\partial}{\partial y'}\right).$$

By integrating the equations of hydrodynamics in the form of conservation laws over an infinitely small region near a discontinuity one finds

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$$\varepsilon(p,\rho)-\varepsilon_0(p_0,\rho_0)=\frac{1}{2}(p+p_0)(\rho_0^{-1}-\rho^{-1}), \tag{1.1}$$
$$p=p_0+jv,\quad v=j(\rho_0^{-1}-\rho^{-1}),$$
$$j=\rho_0\dot{\Xi}\cos\theta,\quad v_x=v\cos\theta.$$

In the case of an ideal gas, by setting $\varepsilon=p/\rho(\gamma-1)$ and after simple algebraic transformations, one obtains from the system (1.1) the following relations at the front:

$$\frac{\rho_0}{\rho}=\frac{\gamma-1}{\gamma+1}\left(1+\frac{2M^{-2}}{(\gamma-1)\cos^2\theta}\right), \tag{1.2}$$
$$v_x=\frac{2\dot{\Xi}}{\gamma+1}(\cos^2\theta-M^{-2}),$$
$$p=\frac{2}{\gamma+1}\rho_0\dot{\Xi}^2\left(\cos^2\theta-\frac{\gamma-1}{2\gamma}M^{-2}\right),$$

where $M=\dot{\Xi}/c_0$; $c_0=(\gamma p_0/\rho_0)^{1/2}$, and for $\theta=0$ the last relations become familiar; this follows from the Hugoniot adiabat. However, in the general case one can see that the relations (1.2) correspond to a shock polar for the portion of the slanting front or to the Buseman conditions expressed in the laboratory reference system, that is, in a system in which an unperturbed gas remains at rest. The planes $\theta=M^{-1}$ determine the maximal slope of slanting fronts, these planes containing the Mach lines for perturbations which start at the front and are propagated to the unperturbed gas [11].

If spontaneous front deflections are present or its deformations due to some random variations of the unperturbed density $\delta\rho_0$, the hydrodynamic functions differ downward the stream from the unperturbed values by the quantities $\delta\rho$, δv, δp, and the front velocity differs by $\delta u=\dot{\Xi}-u(X+\xi)=\dot{\xi}+u(X)-u(X+\xi)$. Small front perturbations which are characterized by the lengthwise (in the propagation direction) and crosswise wave numbers k_x, k_y, so that the quasiclassical approximation is valid for the lengthwise (parallel to x) wave motions and the front distortions are sufficiently smooth, are considered.

By varying the boundary conditions (1.2) in the case of a strong shock wave, a system of equations is obtained in the first approximation for the front deflection:

$$\delta v_x=\frac{2}{\gamma+1}\left(\delta u+\frac{\gamma-1}{2}\delta v_x^0\right), \tag{1.3}$$
$$\delta\rho=\frac{\gamma+1}{\gamma-1}\delta\rho_0,$$
$$\delta p=\frac{2}{\gamma+1}\rho_0u^2\left(\frac{\delta\rho_0}{\rho_0}+2\frac{\delta u}{u}-2\frac{\delta v_x^0}{u}\right),$$

where a change $\delta v_x\to\delta v_x-\delta v_x^0$ was carried out for velocity variation with a possible random velocity of displacement δv_x^0 of the medium upstream taken into account.

It is now convenient to consider all downstream thermodynamic functions as functions of pressure and entropy. Correspondingly, any small front distortion is accompanied by downstream perturbations in the gas of two kinds – entropy or sonic. In the latter form of motion the amplitudes of velocity and pressure are connected by the standard relation for sound (in the quasiclassical approximation). As regards the density perturbations, the latter is equal to the sum of the perturbations due to the above two forms of motion, although it is not advisable at this point to separate the density change into its two components. No other independent modes of eigenoscillations and, in particular, no "surface ones" exist which would be analogous insofar as they also take place for a tangential discontinuity in the case of a shock front. Indeed, if one seeks the solution for these modes in the form of δp, $\delta v_{x,y}\sim\exp(\varkappa x+iky-i\omega t)$ in the domain $x<0$, that is, downstream, then it follows from the linearization of the hydrodynamics equations that $(\varkappa v-i\omega)\delta v_x=-\varkappa\delta p/\rho$; $(\omega+i\varkappa v)\delta v_y=k\delta p/\rho$. Moreover, if $\delta p\neq 0$, then $\delta v_y\neq 0$, which contradicts the continuity of the tangential component of the velocity at the front; upstream from the front only an unperturbed gas can be found, but the condition $\delta p=0$, $\delta v_x\neq 0$ is invalid for real $\varkappa$.

A sound wave arriving from the opposite direction is now considered falling on the front with relative changes of density and velocity given, respectively, by $\delta\rho_0/\rho_0=\delta v_0/c_0\sim\exp(-ik_xx+i\omega t)$. If the medium is entropic, that is, if $p_0\sim\rho_0^{\nu}$, then by the classical approximation one has the amplitude $\delta\rho_0\sim\rho_0^{-(\nu-1)}$, where $\nu=(3\gamma-1)/4$, which also agrees with the conservation law for flows of sound velocity $\rho_0c_0\delta v_0^2=\text{const}$ [12]. The sonic wave refracted at the front of a strong shock wave is propagated orthogonally to the front with an accuracy up to M^{-1} even if the encounter is at an angle [13]. The latter indicates that in the case of a weak front deflection one has $\delta p = -\rho c \delta v_x$ in the quasiclassical approximation. Substituting the latter into the last equation of the system (1.3) and solving it for the perturbations of the front velocity, one obtains approximately (neglecting quantities of the order of M^{-1})

$$\delta u \equiv \dot{\xi} - \xi du/dX = -\lambda_0 u \delta\rho_0/\rho_0.$$

Now one integrates the latter equation in the adopted approximation; the front displacement is thus obtained as a function of the coordinate of the unperturbed front:

$$\xi(X) = \xi_0 \frac{u}{u_0} + \frac{\lambda_0 \delta v_{00}}{ik_x c_{00}} \left[\left(\frac{\rho_{00}}{\rho_0}\right)^{\nu} \exp\left\{- ik_x (X - X_0) + i\omega \int_{X_0}^{X} \frac{dx}{u(x)}\right\} - \frac{u}{u_0}\right], \tag{1.4}$$

where $u = u(X)$; $\rho_0 = \rho_0(X)$; $\rho_{00} = \rho_0(X_0)$; $\delta v_{00} = \delta v_0(X_0)$; ξ_0 is the initial front displacement. If there is no sonic wave, then spontaneous displacements and deflections of the front grow with time according to the rule $\xi \sim u$ [5, 6]. For example, for propagation in a medium with exponentially decreasing density one has $X \sim \ln \rho_0$; $\xi \sim \rho_0^{-\lambda_0}$. The incidence with the front of a sonic wave induces additional front displacements of an oscillatory kind which for $\gamma = 5/3$; $\lambda_0 = 0.2$; $\nu = 1$ grow more rapidly than the spontaneous ones. Deviations from constant density can be regarded as perturbations (then $\delta u = \xi$). In this case the application of the formulas (1.3) directly yields the Chisnell–Whitham relation ($u \sim \rho_0^{-\lambda_0}$). Since this conclusion is based on the quasiclassical approximation, it obviously points to the relatively small part played by long-wave perturbations.

The motion of the front is now considered in a homogeneous and, on the average, weakly turbulent medium, so that the upstream motions of the medium can be represented as a superposition of chaotically distributed sonic waves. The front displacement in the field of one such wave is determined by using the formula (1.4):

$$\xi(y, t) = -\frac{\lambda_0}{\cos(k, x)}[\Delta r(y, X) - \Delta r(y, X_0)], \tag{1.5}$$

where Δr is the displacement of the gas particles in a sound wave. The displacement $\xi_s = \Delta r/\cos(k, x)$ whose projection on the direction of the wave vector is equal to the actual displacement of the particles is now introduced into our considerations. By multiplying the relation (1.5) by $\xi^*(y', t')$, setting $\Delta r(X(t_0)) = 0$, and averaging over the ensemble of waves, one obtains

$$\overline{\xi(y', t')\xi(y, t)} = \lambda_0^2 [K(y' - y; X(t') - X(t)) - K(y' - y; X(t_0) - X(t))],$$

where $K(y, x) = \overline{\xi_s^*(y, t)\xi_s(0, 0)}$ is the correlation function of the front displacements.

To compute the displacements in the second approximation, that is, the nonlinear correction ξ_2 by varying the boundary conditions (1.2), one must again set $\delta p = -\rho c \delta v_2$ and take into account that $\delta u = \dot{\xi}_2 - \xi_2 du/dX - (\xi_1^2/2)d^2u/dX^2$. The following differential equation is then obtained:

$$\frac{d\xi_2}{dX} - \xi_2 \frac{d \ln u}{dX} = \frac{\xi_0^2 u}{2u_0^2}\frac{d^2 u}{dX^2} + \theta^2(1 - \lambda_0),$$

and in the case of exponential density one has

$$\rho_0 \sim \exp(-X/l), \quad \xi_2 = \frac{\lambda_0 \xi_0^2}{2l}(e^{2z} - e^z) + \theta^2(\lambda_0^{-1} - 1)(e^z - 1), \quad z = \lambda_0 X/l,$$

where $\xi_2 < \xi_1$, if $\lambda_0 \xi_0 < l$ (the inequality is also valid for $\xi_0 \leq l$).

If in the first approximation the growth of front displacements from the equilibrium position takes place symmetrically with respect to lagging behind or outstripped displacements, then asymmetry arises if nonlinear corrections are used: The outstripping takes place relatively more rapidly than the lagging behind of the front elements, this effect being more substantial the greater the amplitude of the displacements and the front deflection.

§2. The motion of a medium with an arbitrary state equation is considered. In this case it is advisable to modify the procedure for varying the boundary conditions. Having differentiated the first equation of the system (1.1) and employed the principles of thermodynamics, one finds the equation connecting the increments of pressure and of density on the Hugoniot adiabat:

$$\left(\frac{\partial p}{\partial \rho}\right)_H = \frac{c^2[1 - \Gamma(p - p_0)/2\rho c^2]}{1 - \Gamma(\rho - \rho_0)/2\rho_0}, \quad \Gamma = \frac{(\partial p/\partial T)_\rho}{\rho C_V},$$

where Γ is the Grüneisen constant, and $c^2 = (\partial p/\partial \rho)_S$. If one now employs the remaining equations of the system (1.1), one obtains in a similar manner

$$(\partial\rho/\partial v)_H = 2j[1 + (j^2/\rho^2)(\partial\rho/\partial p)_H]^{-1}. \tag{2.1}$$

The motion of a strong shock wave when the upstream gas pressure can be ignored, the variation of the gas pressure downstream being $\delta p = (\partial p/\partial v)_H \delta v + (\partial p/\partial \rho_0)\delta\rho_0$, is now considered. The derivative which appears in the latter expression can be determined for a constant gas velocity from the system (1.1), $(\partial p/\partial \rho_0)_V = j^2\rho_0^{-2}[1-(\partial\rho^{-1}/\partial\rho_0^{-1})_V]$. On the other hand, on has, as before, $\delta p = -\rho c \delta v$. By varying directly the second relation of (1.2) one finds $\delta p = \rho_0 u_n \delta v + \rho_0 v \delta u_n + v u_n \delta\rho_0$, where $u_n = u\cos\theta$ is the rate of its displacement normal to the surface. By comparing all three pressure variations one finally obtains

$$\frac{\delta u_n}{u_n} = -\frac{\delta\rho_0}{\rho_0}\left[1 - \frac{u_n(\rho c + \rho_0 u)}{v(\rho c + (\partial p/\partial v)_H)}\left(1 - \frac{\partial\rho^{-1}}{\partial\rho_0^{-1}}\right)\right]. \tag{2.2}$$

The above formula is obviously a generalization of the Chisnell–Whitham formula not only in the case of an arbitrary state equation, but also for a front in motion at an arbitrary angle to the direction of density reduction.

In a medium which is homogeneous in density the variation $\delta\rho_0$ must be set equal to zero; since the equations are homogeneous, all other variations must also vanish, excluding only the case in which the denominator in the square brackets in (2.2) vanishes, that is, $\rho c + (\partial p/\partial v)_H = 0$. The latter can, with the aid of the relation (2.1), be transformed into

$$f(m) = 1 + m - am^2(1-m)/(1-am^2) = 0,$$

where $a = (\Gamma/2)(\rho/\rho_0 - 1)$; $m = |u - v|/c$ is the downstream Mach number.

For compression waves, if the inequality $0 < m < 1$ [14] is valid, the above condition can hold provided $a > 1$. For an ideal gas one has $a = 1$, $f > 0$ and the front remains stable. A condition was obtained in [8] which ensures that the function f(m) crosses the zero value for an unstable front in a single-phase medium, though in a somewhat different manner. This condition is satisfied only for a special form of the Hugoniot adiabat.

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